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# DUNKL ANALYSIS AND APPLICATION TO INVERSE SOURCE PROBLEMS

## ABSTRACT

of the PhD thesis for the degree of
doctor of Philosophy (PhD) in the specialty
«6D060100-Mathematics»

**The relevance of the research topic.** In this dissertation, we investigate pseudo-differential operators generated by the Dunkl operators and consider inverse source problems for the parabolic and pseudo-parabolic equations.

Pseudo-differential operators generated by the Dunkl operators were first studied by A. Dachraoui in 2001 and several boundedness results were obtained for these operators in classical Schwarz spaces and Sobolev-type spaces. This is followed by several papers where the $L^2$ and $L^p$-boundedness of these operators were studied, but the boundedness of amplitude, transpose and adjoint operators in Schwarz spaces were not studied.

In this dissertation, we prove the boundedness of pseudo-differential, amplitude, transpose and adjoint operators with classical symbols, generated by the Dunkl operators, in Schwarz spaces. Inverse source problems for parabolic and pseudo-parabolic equations with fractional derivatives of Caputo and Hilfer in time, which had not been considered before, were also studied. It was shown that these problems are correctly solvable in the Hadamard sense and explicit form of solutions to these problems were obtained.

**The aim of the PhD thesis** is to develop the theory of pseudo-differential operators generated by the Dunkl operators and to study inverse source problems for the parabolic and pseudo-parabolic equations.

**To achieve the aim of the dissertation, the main objectives** of the following research are considered:

-Prove that the pseudo-differential operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

-Prove that the amplitude operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

-Prove that the transpose operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

-Prove that the adjoint operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

-Define the kernels of the pseudo-differential operators generated by the Dunkl operators and show that the kernels are smooth functions.

-Show the correct solvability of the inverse source problem for the heat equation with the Caputo fractional derivative in time, generated by the Dunkl operator.

-Show the correct solvability of the inverse source problem for the pseudo-parabolic equation with the Caputo fractional derivative in time, generated by the Dunkl operator.

-Show the correct solvability of the inverse source problem for the heat equation with the Hilfer fractional derivative in time, generated by the Dunkl operator.

**The object of the PhD thesis** is Dunkl operators and pseudo-differential operators generated by the Dunkl operators.

**The methods of scientific research.** The dissertation uses the methods of the theory of pseudo-differential operators, the theory of partial differential equations, the theory of functions and the theory of special functions.

**Scientific novelty of the work.** The problems that considered in this dissertation are new.

The following new tasks are presented in the dissertation as an application of Dunkl analysis:

We will consider the following problem

$$\begin{cases} \mathcal{D}^{\gamma}_{0^+,t}u(t,x) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), & (t,x) \in Q_T, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}, \\ u(T,x) = \psi(x), & x \in \mathbb{R}, \end{cases}$$

where $Q_T := \{(t,x):\ 0 < t < T < +\infty,\ x \in \mathbb{R}\}$, $\mathcal{D}^{\gamma}_{0^+,t}$ is the Caputo fractional derivative, $D^2_{\alpha,x}$ is the Dunkl Laplacian, and $\varphi, \psi$ are given suitable functions. In the dissertation, we showed that the above problem has a unique pair of solutions $(u, f)$ in Sobolev-type spaces and showed an explicit form of the solution. Then we considered the following problem, as a continuation of the above problem,

$$\begin{cases} \mathcal{D}^{\gamma}_{0^+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), (t,x) \in Q_T, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}, \\ u(T,x) = \psi(x), & x \in \mathbb{R}, \end{cases}$$

where $a$ is positive number, if $a$ is equal to zero, then we obtain the first problem. Here we have shown that the problem has a unique pair of solutions $(u, f)$ in Sobolev type spaces and found an explicit form of the solution. At the end, we considered the problem

$$\begin{cases} \mathcal{D}_{0^+,t}^{(\gamma_1,\gamma_2)s} u(t,x) - aD_{\alpha,x}^2 u(t,x) = p(t)f(x), & (t,x) \in Q_T, \\ \lim\limits_{t\to 0+} I_{0^+,t}^{1-\eta} u(t,x) = \varphi(x), & x \in \mathbb{R}, \\ u(T,x) = \psi(x), & x \in \mathbb{R}, \end{cases}$$

where $\mathcal{D}_{0^+,t}^{(\gamma_1,\gamma_2)s}$ is the Hilfer fractional derivative, $I_{0^+,t}^{1-\eta}$ is the Riemann-Liouville fractional integral, and $p$ is the given function. Here, we have also shown that the problem has a unique pair of solutions $(u,f)$ in Sobolev-type spaces and found an explicit form of the solution.

**Theoretical and practical significance of the results.** The research on the topic is mainly theoretical and fundamental. Their scientific significance is due precisely to the deep level of fundamentality of the results obtained.

**Publications.** 7 works have been published (2 in journals indexed by Scopus, and 4 in a journal recommended by the Committee for Quality Assurance in the Field of Science and Higher Education MSHE RK).

Publication in the high-ranking scientific journals

1 B. Bekbolat, D. Serikbaev, N. Tokmagambetov. Direct and inverse problems for time-fractional heat equation generated by Dunkl operator // Journal of Inverse and Ill-Posed Problems, V. 31, № 3, 393–408, 2023. Indexed by Scopus and Web of Science, Scopus SJR 2022: 0.428 (Q2), CiteScore 2022: 2.4, Web of Science Impact factor: 1(Q2)

2 B. Bekbolat, A. Kassymov, N. Tokmagambetov. Blow-up of solutions of nonlinear heat equation with hypoelliptic operators on graded Lie groups // Complex Analysis and Operator Theory. V. 13, № 7, 3347-3357, 2019. Indexed by Scopus and Web of Science, Scopus SJR 2018: 0.459 (Q3), CiteScore 2019: 1.3, Web of Science Impact factor: 0.8(Q3)

CQAFSHE

1 B. Bekbolat, N. Tokmagambetov. Well-posedness results for the wave equation generated by the Bessel operator // Bulletin of the Karaganda University, V. 101, № 1, 11-16, 2021.

2 B. Bekbolat, D. B. Nurakhmetov, N. Tokmagambetov, G. H. Aimal Rasa. On the minimality of systems of root functions of the Laplace operator in the punctured domain // News of the national academy of sciences of the republic of Kazakhstan, Physico-mathematical series. V. 4, № 326, 92-109, 2019.

3 B. Bekbolat, B. Kanguzhin, N. Tokmagambetov. To the question of a multipoint mixed boundary value problem for a wave equation // News of the

national academy of sciences of the republic of Kazakhstan, Physico-mathematical series. V. 4, № 326, 76-82, 2019.

4 B. Bekbolat, N. Tokmagambetov. On a boundedness result of non-toroidal pseudo differential operators // International Journal of Mathematics and Physics. V. 9, № 2, P. 50-55, 2018.

Kazakh local journal

1 B. Bekbolat, N. Tokmagambetov. Cauchy problem for the Jacobi fractional heat equation // Kazakh Mathematical Journal, V. 21, № 3, 16-26, 2021.

**The structure and scope of the thesis.** The dissertation work consists of a title page, a table of contents, an introduction, three sections, a conclusion and a list of references. The total volume of the dissertation is 109 pages with 99 references to literature.

**The main content of the thesis.** The introduction contains the relevance of the research topic, goals and objectives, the main provisions for the defense of the dissertation, the object and subject of the research, research methods, novelty and theoretical and practical significance of the research, the connection of the PhD thesis with other research papers, the approbation of the work, the author's publications, the scope and structure of the dissertation and content.

In the first chapter, we collect some basic results in the Dunkl analysis and fractional analysis. We define the Dunkl operator $D_\alpha$ and the Dunkl Laplacian $D_\alpha^2$ on the suitable spaces and consider properties of the Dunkl operator. We define the Dunkl kernel $E_\alpha(x,y)$ as a unique solution of the initial value problem generated by the Dunkl operator. Then we obtain series and Poisson integral representations of the Dunkl kernel. We prove that the Dunkl kernel $E_\alpha(x,y)$ does not have zeros for all $x, y \in \mathbb{R}$. Then we define Dunkl and inverse Dunkl transforms and study their properties. After we define the Dunkl transform on tempered distributions and prove that it is a continuous linear transformation. Also, we give Taylor series generated by the Dunkl operator, as a part of Dunkl analysis.

In the second chapter, we consider pseudo-differential operators generated by the Dunkl operator. Some boundedness results for these operators were already known in the literature. We define amplitude, adjoint and transpose operators and prove that pseudo-differential, amplitude, adjoint and transpose operators are linear transformations on the Schwartz spaces. We also define pseudo-differential operators on tempered distributions and prove that it is a continuous linear transformation. Then we study properties of the distributional and convolution kernels of the pseudo-differential operators. In particular, we prove Schur's lemma. We obtain some boundedness results on spaces $L(\mathbb{R}, d\mu_\alpha)$ for the pseudo-differential operators and composition of the pseudo-differential operators, under certain assumptions.

In the last chapter, we study inverse source problems for Dunkl-heat and Dunkl-pseudo-parabolic equations with Caputo and Hilfer fractional differential operators. For this inverse source problems, we prove well-posedness results in the sense of Hadamard. First, we consider direct problems and establish the unique existence of a generalized solution. Then we consider inverse source problems and define pair of solutions in suitable spaces. We use classical Fourier method. After we establish stability results, which means that the solution of the inverse source problems continuously depends on given data. Additionally, we consider some examples to give an illustration of our analysis.